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Technical Note

Explicit analytical solutions of incompressible unsteady 2-D laminar flow with heat transfer

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1. Introduction

Analytical solutions of steady incompressible potential flow played a key role in the early development of fluid mechanics. It is the same for the heat conduction. However, the governing equations of unsteady viscous flow are nonlinear, hence, it is difficult to obtain analytical solutions of such flow. According to the knowledge of the authors, no new explicit analytical solution of unsteady viscous flow with heat transfer has been found in the open literature for many years. In order to fill in the gaps in the field of unsteady viscous fluid mechanics and heat transfer, it is meaningful in theory to find out some new ways for analytical solutions, which may be useful to check the accuracy, convergence and effectiveness of various numerical computation methods. For example, several analytical solutions which can simulate the 3-D potential flow in turbomachine cascades were given by the first author [1]. These solutions have been used successfully by some numerical calculation scientists to check their computational methods and computer codes [1–4]. In addition, the authors recently presented some explicit analytical solutions of unsteady compressible flow and heat transfer [5–10]. Several new explicit analytical solutions of unsteady viscous flow with heat transfer are given in this paper to serve as the benchmark solutions for numerical calculations. The derivation procedure in this paper is mainly based on the method of separation variables with addition applied by the authors [7–9]. This method is to separate an unknown function $f(x,y)$ with assumption $f = X(x) + Y(y)$ instead of $f = X(x) \cdot Y(y)$, its correctness for a given analytical solution, can be proven easily by substituting it into the governing equations.

2. Governing equations

The governing equations for constant kinematic viscosity and thermal diffusivity, unsteady 2-D incompressible laminar flow with heat transfer can be presented as follows (neglecting gravity, radiation and internal heat source) [11]:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial (p/\rho)}{\partial x} + \upsilon\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \tag{1b}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{\partial (p/\rho)}{\partial y} + \upsilon\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right), \tag{1c}$$

$$\begin{aligned}\frac{\partial \theta}{\partial t} + u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} &= a\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2}\right) + \frac{\upsilon}{C_p}\left[\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 \right.\\ &\quad \left. + 2\left(\frac{\partial u}{\partial x}\right)^2 + 2\left(\frac{\partial v}{\partial y}\right)^2\right].\end{aligned} \tag{1d}$$

With boundary layer assumption, the governing equations can be simplified as:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2a}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{\partial (p/\rho)}{\partial x} + v\frac{\partial^2 u}{\partial y^2}, \tag{2b}$$

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Nomenclature

a	thermal diffusivity
C	constant
C_p	specific heat
p	pressure
t	time
u	velocity component in x direction
v	velocity component in y direction
x	abscissa
y	coordinate
θ	temperature
υ	kinematic viscosity
ρ	density
Subscripts	
$0, 1, 2, 3, \ldots$	different constants
∞	free stream

$$\frac{\partial\theta}{\partial t} + u\frac{\partial\theta}{\partial x} + v\frac{\partial\theta}{\partial y} = a\frac{\partial^2\theta}{\partial y^2} + \frac{\upsilon}{C_p}\left(\frac{\partial u}{\partial y}\right)^2. \tag{2c}$$

3. Solutions for flow between two parallel porous plates moving with different velocities

An algebraically explicit analytical solution of the governing equation set (1a)–(1d) of constant coefficient unsteady 2-D incompressible laminar flow with heat transfer can be derived mainly with the method of separating variables with addition as follows.

We assume that:

$$u = f(x) + g(y) + d(t), \tag{3}$$

$$v = h(x) + j(y) + b(t), \tag{4}$$

$$p/\rho = X(x) + Y(y) + T(y). \tag{5}$$

Then, the following relation can be derived from governing equations (1a) with separation of variables: $f'(x) = C_9 = -j'(y)$, i.e.

$$f(x) = C_9 x + C_{10} \tag{6}$$

and

$$j(y) = C_8 - C_9 y. \tag{7}$$

Substituting Eqs. (3)–(7) into governing equations (1b) and (1c), it is obtained:

$$d'(t) + C_9[C_9 x + C_{10} + g(y) + d(t)] + g'(y) \times [h(x) + C_8 - C_9 y + b(t)] = \upsilon g''(y) - X'(x), \tag{8}$$

$$b'(t) + h'(x)[C_9 x + C_{10} + g(y) + d(t)] - C_9[h(x) + C_8 - C_9 y + b(t)] = \upsilon h''(x) - Y'(y) \tag{9}$$

Rearranging Eq. (8), it is deduced that

$$C_9 d(t) + d'(t) + g'(y)b(t) + C_9^2 x + C_9 C_{10} + X'(x) + g'(y)h(x) = -C_9 g(y) + (C_9 y - C_8)g'(y) + \upsilon g''(y). \tag{10}$$

It can be understood from Eq. (10) that the separation of variables is able to be accomplished further when:

$$\begin{aligned} &d'(t) = -X'(x) = \text{Const} = C_0,\\ &h(x) = \text{Const} = C_1 - C_8 - C_{11},\\ &b(t) = \text{Const} = C_{11},\\ &C_9 = 0 \end{aligned} \tag{11}$$

or $d(t) = 0 = b(t)$

$$\begin{aligned} &h(x) = \text{Const} = C_1,\\ &Y'(x) = -C_4,\\ &C_8 = 0 = C_9. \end{aligned} \tag{12}$$

Case 1. Substituting Eq. (11) into Eqs. (8) and (9), it is easy to have

$$g(y) = C_2 e^{C_1 y/\upsilon} + C_3, \tag{13}$$

$$Y(y) = 0 \tag{14}$$

and

$$T(t) = \text{arbitrary function of } t. \tag{15}$$

Combining Eqs. (3)–(7), (11) and (13)–(15), one of the solutions of the dynamic relation (1b) and (1c) can be given as follows:

$$u = C_2 e^{C_1 y/\upsilon} + C_0 t + C_3, \tag{16a}$$

$$v = C_1, \tag{16b}$$

$$p/\rho = p_0/\rho - C_0 x + T(t). \tag{16c}$$

For the energy equation (1d), assume that:

$$\theta = X_1(x) + Y_1(y) + T_1(t) \quad \text{and} \quad X_1(x) = C_4 x + C_5,$$

then we are able to derive the explicit analytical solution of θ with the same procedure as mentioned above. The final result will be

$$\theta = \frac{C_2 C_4 \upsilon}{C_1^2(a/\upsilon - 1)} e^{C_1 y/\upsilon} - \frac{C_2^2 \upsilon}{2C_p(2a - \upsilon)} e^{2C_1 y/\upsilon} + \frac{C_5 a}{C_1} e^{C_1 y/a} - \frac{(C_3 + C_7)C_4}{C_1} y + C_4\left(x + C_7 t - \frac{C_0}{2}t^2\right) + C_6. \tag{16d}$$

The physical description of this solution for steady velocity case ($C_0 = 0, C_7 = 0$ and $T(t) = 0$) is given in Fig. 1. There are two infinite porous plates parallel to x abscissa moving along the abscissa direction with different speeds which are given by Eq. (16a). The flow field between the porous plates is described by Eqs. (16a)–(16d). The velocity of fluid flow, u, is only a function of y and is given by Eq. (16a) also (it is assumed in Fig. 1 that the constants C_1, C_2 and C_3 are all positives). When fluid is injected from the lower porous plate into the flow field between two porous plates with a uniform constant velocity component in y direction $v = C_1$, and ejected to the upper porous plate with same velocity, v, the velocity component in y direction in the flow field would be constant, C_1, too. The temperature varies along the x direction with linear relation and mainly varies along y direction and its variation curve is somewhat similar to the u curve shown in Fig. 1. Heat transfers from upper plate to the flow between two plates as well as from the flow to the lower plate. The boundary conditions can be obtained with Eq. (16d).

For the unsteady velocity case, besides the velocity component in x direction described previously, both two porous plates and the fluid between the plates accelerate in x direction with acceleration C_0; but the velocity component in y direction is still constant: $v = C_1$. The pressure changes along x direction with a constant negative pressure gradient $C_0\rho$ to maintain the acceleration. The temperature variation is mathematically 3-D $\theta(t, x, y)$. Actually $T(t)$ is an arbitrary function and it does not influence the kinematic and temperature condition of the flow field.

The expression of absolute coordinates streamlines in a fixed time can be obtained by integrating $\mathrm{d}y/\mathrm{d}x = v/u$ as follows:

$$x = \frac{C_2 \upsilon}{C_1^2} \mathrm{e}^{C_1 y/\upsilon} + \frac{C_0 t + C_3}{C_1} y + C_8. \tag{17}$$

When the flow is steady ($C_0 = 0$), the streamlines in the absolute coordinates are shown in Fig. 2.

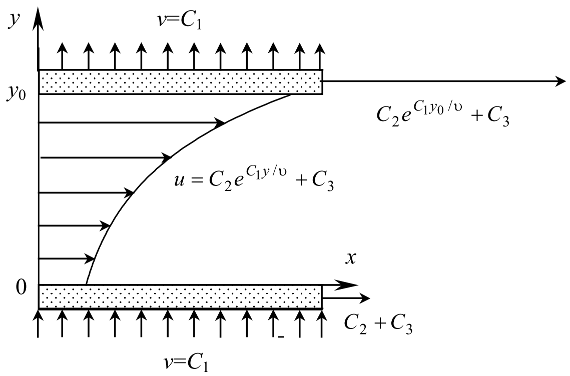

Fig. 1. The flow condition of steady case of Eqs. (16a)–(16d).

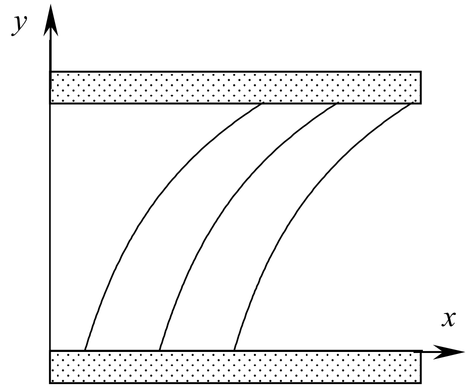

Fig. 2. The steamline of steady case of Eqs. (16a)–(16d).

Case 2. Another algebraically explicit analytical solution can be derived similarly with Eq. (12) as follows:

$$u = C_2 \mathrm{e}^{C_1 y/v} + \frac{C_4}{C_1} y + C_3, \tag{18a}$$

$$v = C_1, \tag{18b}$$

$$p/\rho = p_0/\rho - C_4 x + T(t), \tag{18c}$$

$$\begin{aligned}\theta = {} & \frac{C_2 \upsilon^2 (C_5 - 2C_4/C_\mathrm{p})}{C_1^2 (a - \upsilon)} \mathrm{e}^{C_1 y/\upsilon} \\ & - \frac{C_2^2 \upsilon}{2C_\mathrm{p}(2a - \upsilon)} \mathrm{e}^{2C_1 y/\upsilon} + C_8 \mathrm{e}^{C_1 y/a} - \frac{C_4 C_5}{2C_1^2} y^2 \\ & - \left(\frac{C_3 C_5}{C_1} + \frac{C_6}{C_1} + \frac{a C_4 C_5}{C_1^3} - \frac{C_4^2 \upsilon}{C_1^3 C_\mathrm{p}} \right) y \\ & + C_5 x + C_6 t + C_7. \end{aligned} \tag{18d}$$

This solution is very similar to that expressed by Eqs. (16a)–(16d). The main exception is that the velocity component, u, is steady now in this solution. In addition, the temperature function is a little bit more complicated. Then, some physical descriptions of previous solution, as described by Figs. 1 and 2 can be used for this solution also; the only main difference is that the curves of $u(y)$ in Fig. 1 and streamline curves in Fig. 2 will be more inclined when all constants C_i are positive.

Although the solutions expressed by Eqs. (16a)–(16d) and Eqs. (18a)–(18d) are very similar, they are different solutions and cannot be derived from each other.

4. Extended Couette flow

A very simple linear velocity and temperature variation solution of Eqs. (1a)–(1d) can be deduced as:

$$u = C_4 t + C_1(x - y) + C_2, \tag{19a}$$

$$v = C_4 t + C_1(x - y) + C_3, \tag{19b}$$

$$p/\rho = [C_1(C_3 - C_2) - C_4](x + y) + C_5 + T(t), \tag{19c}$$

$$\theta = \left[(C_3 - C_2)C_6 + \frac{4\upsilon}{C_p}C_1^2\right]t + C_6 x - C_6 y + C_7, \tag{19d}$$

where $T(t)$ is an arbitrary function of time.

The constant C_3 in Eqs. (19a)–(19d) has to be unequal to C_2, otherwise $u \equiv v$ and then it is an unsteady flow with uniform velocity distribution. Considering the Eqs. (19a)–(19c), this solution is actually an Euler flow solution to satisfy the boundary conditions demanded by viscous laminar flow.

It is able to satisfy the demand by turning the coordinates by 45°, and then Eqs. (19a)–(19d) becomes:

$$u = C_4 t + C_1 y + C_2, \tag{20a}$$

$$v = C_3, \tag{20b}$$

$$p/\rho = C_5 - (C_1 C_3 + C_4)x + T(t), \tag{20c}$$

$$\theta = \left(C_3 C_6 + \frac{\upsilon}{C_p}C_1^2\right)t - C_6 y + C_7. \tag{20d}$$

In order to explain the physical condition of Eqs. (20a)–(20d), we begin with the simplest case of Eqs. (20a)–(20d) with $C_2 = 0,\ C_3 = 0,\ C_4 = 0$ and $T(t) = 0$

$$u = C_1 y, \tag{21a}$$

$$v = 0, \tag{21b}$$

$$p/\rho = C_5, \tag{21c}$$

$$\theta = \frac{\upsilon}{C_p}C_1^2 t - C_6 y + C_7. \tag{21d}$$

It is a standard simple Couette flow between two parallel solid plates. The first term in Eq. (21d) represents the temperature increase due to dissipation; the second term represents constant heat transfer across the channel. The flow is driven by the moving plate and having no need of pressure drop in the flow direction.

When $C_3 = 0$ only, Eqs. (20a)–(20d) become:

$$u = C_4 t + C_1 y + C_2, \tag{22a}$$

$$v = 0, \tag{22b}$$

$$p/\rho = C_5 - C_4 x + T(t), \tag{22c}$$

$$\theta = \frac{\upsilon}{C_p}C_1^2 t - C_6 y + C_7. \tag{22d}$$

This is an extended Couette flow, the boundary solid plates of Couette flow accelerate with an acceleration C_4. The velocity distribution is still linear across the flow. The second term of Eq. (22c) represents the pressure drop for accelerating the flow.

When $C_3 \neq 0$ (Eqs. (20a)–(20d)), it represents the extended Couette flow with fluid injection and ejection through the porous boundary plates. The pressure drop increases due to existence of cross-velocity $v = C_3$. The temperature increase with time is higher due to the cross-velocity $v = C_3$ carries some heat (internal energy) into the flow.

5. Degenerative solution describing boundary layer suction

The solution Eqs. (16a)–(16d) can become a steady boundary layer suction solution of governing equations (2a)–(2c) when the constants in Eqs. (16a)–(16d) satisfy the following conditions: $C_0 = 0,\ C_1 = -C_1',\ C_2 = -\upsilon/(C_1'C_2'),\ C_3 = \upsilon/(C_1'C_2'),\ T(t) = 0,\ C_4 = 0,\ C_5 = -1/C_3',\ C_6 = \theta_\infty$.

Neglecting the superscript 'prime' in the new case, and assuming all constants are positive, Eqs. (16a)–(16d) can be rewritten as follows:

$$u = \frac{\upsilon}{C_1 C_2}(1 - e^{-C_1 y/\upsilon}), \tag{23a}$$

$$v = -C_1, \tag{23b}$$

$$\theta = \theta_\infty - \frac{a}{C_1 C_3}e^{-C_1 y/a} - \frac{\upsilon^3}{2C_1^2 C_2^2 C_p(2a - \upsilon)}e^{-2C_1 y/\upsilon}. \tag{23c}$$

It is exactly the boundary layer suction solution [11].

6. Summary

Some new explicit analytical solutions of unsteady 2-D laminar flow with heat transfer and flow injection and ejection through the boundary porous plates are given. According to the knowledge of the authors, no such analytical solutions have been published in the open literature. These solutions are valuable to the theory of viscous flow and heat transfer, especially to the computational fluid dynamics and computational heat transfer as the benchmark solutions to check the numerical solutions and to develop the numerical computation approaches such as the differencing schemes, grid generation methods and so forth.

In addition, the analytical solution to steady boundary layer with suction can be derived as a special case of the abovementioned solutions.

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